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A Unified Probabilistic Approach of Tunisian Stock Market Cycle: Nonlinearity, Turning Points and Duration-Dependence

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Abstract

In this paper, we have two goals. First, we try to identify the stock market states and outline their statistical properties by using Multi-states Duration-Dependence Markov-switching models. Results show that the three-state model outperforms other models. An application to Tunisian stock market reveals that there exists three different states and each state represents different features of Tunisian stock market. Second, we construct a turning index based on the smoothed probabilities in order to identify the different Tunisian market cycle phases. The relevance of our index was documented from the synchronization between the values of the turning index and the values of TUNINDEX index return. It is well-adapted in order to account for extreme events.

Key Words: *Duration-Dependence; Risk-Return Trade-off; Tunisian Stock Market; Markov-Switching Model; Turning index.*

Introduction

Identifying the precise moments when the stock market bounces back provides insider information for everyone who pays attention to the short-term evolution of the stock market. Detecting and predicting these turning points allow to alert investor to implement an investment strategy to limit the impact of slowdown or recession phases or otherwise profit from acceleration or expansion phases.

But, when referring to the empirical analysis of cyclical movements, some confusion about the definition of cycle phases and the complex temporal behavior of financial time series are well-documented. In this respect, it has been shown that financial time series of high frequency such as daily or weekly stock series display a range of seeming features common across a wide range of instruments, markets and time periods. These empirical and statistical properties are called stylized facts such as volatility clustering, high kurtosis, long memory, fat-tailed densities, and nonlinear structure in returns. Among these properties, volatility clustering emerges, as reported by Mandelbrot (1963), when “large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes”. As a result, absolute returns or their squares show a positive, significant and slowly decaying autocorrelation function (Cont, 2001). To study this phenomenon, a large class of stochastic models was developed in finance. Especially, GARCH models and stochastic volatility models are mainly appointed to model the volatility clustering (e.g. Engle, 1995; Bollerslev et al., 1992). Many enhancements of classical GARCH models add complexity in order to grasp the volatility of financial markets and capture the clustering volatility phenomenon. In this respect, the GARCH-in Mean (GARCH-M) model proposed by Engle, Lilien and Robins (1987) attempts to model the relationship between the conditional mean return and variance (risk). Formally, the GARCH-M model adds the heteroskedasticity term into mean return equation.

Likewise, a great deal of attention has been paid to the non-linear structure and cyclical behavior of stock returns. To capture and reproduce this feature, Markov-switching models of Hamilton (1989) have become extensively used to highlight regime shifts behavior in both the mean and the variance according to a latent state variable S_t, which takes on a finite number of values. However, the Markov-switching model of Hamilton (1989) does not account for the duration dependence in nonlinear dynamics of the stock returns. The Markov-switching framework was later generalized to allow for time-varying, duration-dependent and seasonally dependent transition probabilities (e.g. Durland and McCurdy, 1994; Chen and Shen, 2006). As such, Chen and Shen (2006) adopt a Markov-switching model with a duration dependence structure in transition probabilities in order to study the duration dependence feature of Taiwan's business cycles. The results prove different periods of contraction and expansion characterizing by (or no) dependence duration. Maheu and McCurdy (2000) used Duration-Dependence Markov-Switching model to detect high return stable state (bull market) and low return-volatile state (bear market). By incorporating the duration dependence, they found that the best profit is maked at the beginning of the bull market and volatility grows over the duration of a bear market. Woodward and Marisetty (2005) argue that the length of time spent in the bear and bull market state may be a primary determinant in order to clarify the risk-return trade-off of risky assets. This reflects the importance of the duration of a particular market condition when measuring risk.

In this context, several researchers retain two-state specification as provided in traditional classifying of stock markets into two regimes labeled bear and bull markets (e.g. Maheu and McCurdy, 2000). For example, Gordon and St. Amour (2000) used two-state Markov-switching model in order to capture the moderate and infrequent movements in risk aversion and the cyclical nature of observed asset prices. Nonetheless, more states could be used to allow a rich description of intrastate dynamics. Guidolin and Timmermann (2005) show that three-state specification captures key features of UK stock and bond returns. Given these different features, the Markov-Switching are intensely used as dating method[1] of the stock market cycles. Under these models, the different states are determined using the marginal transform $\varphi(.)$ such that $\varphi(r_i)=S_{it}(\forall i)$ where S_{it} is 1…..j (j is the number of state).

This study lies in the same perspective and attempts to provide a probabilistic lecture of Tunisian[2] stock market cycle based on a turning index from the estimation of three-state Duration-Dependence Markov-Switching L^{th} order Autoregressive Duration-Dependence GARCH-M (DD(τ)-MS(3)-AR(L)-DD(τ)-GARCH-M(1,1)) model on the TUNINDEX stock index weekly returns during January 07, 1998 till March 29, 2013. From a methodology perspective, the proposed model consist of an extension of Maheu and

[1] The traditional method of identifying the stock market phases are the dating algorithms based on a set of rules for classification such as in Lunde and Timmermann (2004) and Pagan and Sossounov (2003). A drawback of the dating algorithm is that it cannot be used for statistical inference on returns or for investment decisions which require more information from the return distribution (Maheu et al., 2012).

[2] An interesting question is to determine if cyclical regimes are more pronounced in emerging markets. Emerging markets are the more obvious candidates for detecting and identifying changes in cyclical stock market synchronization by virtue to rapid transformations of their financial systems (Candelon et al., 2008) and the transformations instability when facing exceptional difficult conditions. The Tunisian case provides a good example since the country's revolution had the adverse consequences on the economy. During January 2011, the TUNINDEX index is closed with loss of about 13.29% and the trading on Tunisian stock market was halted for two weeks. Since the outbreak of the popular uprising, tourism, Tunisia's largest sources of foreign currency, has fallen by more than 50%. Foreign direct investment has dropped by 20% and more than 80 foreign companies have left the Tunisian economy. Meanwhile, the depreciation of the Tunisian dinar as well as the budget and current account deficits have deeply increased. These hostile national conditions have accompanied by a lack of liquidity and a high cost of external financing due to the downgrading of sovereign debt ratings. We therefore use Tunisian stock market data in our empirical application.

McCurdy (2000)'s model in three directions: (i) we add an additional state in order to highlight the nonlinear dynamics of index returns; (ii) we examine the duration dependence in the conditional mean return, volatility, risk-return trade-off and the transition probabilities and (iii) we take into account concurrently the GARCH and duration dependence effects in the volatility equation.

This paper has the following contents: The model is developed in section 2. In section 3, the financial dataset used for empirical application is presented. The different estimation results are provided in section 4. Section 5 is devoted to exhibit the calculation results of the index turning while the discussion is reported in the sixth section. Concluding remarks are contained in Section 6.

Model Development

The nonlinearities we hope to examine through this modeling encompass asymmetric cycles and time variation in the conditional moments of stock returns. We use a three-state Duration-Dependence Markov-Switching L^{th} order Autoregressive Duration-Dependence GARCH-M (DD(τ)-MS(3)-AR(L)-DD(τ)-GARCH-M(1,1)) model. Duration dependence is emphasized in the conditional mean return, volatility, risk-return trade-off as well as the transition probabilities.

At each time t, the return series is assumed to belong to one of three regimes[3]. Let S_t^* denote a latent variable which takes the values {1, 2, 3}. The transition dynamics between the three regimes is described by a homogeneous semi-Markov process. According to this process, transition intensities depend on duration which expressed by a latent variable $D_{S_t^*}$. In other words, $D_{S_t^*}$ is the number of successive periods recently spent in the same regime:

$$D_{S_t^*} = \begin{cases} D_{S_{t-1}^*} + 1 & \text{if} \quad S_t^* = S_{t-1}^* \\ 1 & otherwise \end{cases} \tag{1.1}$$

As stated by Maheu and McCurdy (2000), the transition probabilities are parameterized using the logistic function. For each regime i $(i = 1, 2, 3)$, the transition probabilities are specified conditionally on $D_{S_{t-1}^*}$ as follows:

$$p_{ij}^d = \Pr\left(S_t^* = j / S_{t-1}^* = i;\ D_{S_{t-1}^*} = d\right) = \frac{exp\left[\lambda_1^{ij} + \lambda_2^{ij}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right]}{1 + \exp\left[\lambda_1^{ij} + \lambda_2^{ij}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right] + exp\left[\lambda_1^{ik} + \lambda_2^{ik}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right]} \tag{1.2}$$

$$p_{ik}^d = \Pr\left(S_t^* = k / S_{t-1}^* = i;\ D_{S_{t-1}^*} = d\right) = \frac{exp\left[\lambda_1^{ik} + \lambda_2^{ik}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right]}{1 + \exp\left[\lambda_1^{ij} + \lambda_2^{ij}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right] + exp\left[\lambda_1^{ik} + \lambda_2^{ik}\left(dI_{(d\le\tau)} + \tau I_{(d>\tau)}\right)\right]} \tag{1.3}$$

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[3] The choice of the three states is justified a posteriori using the Likelihood-ratio test proposed by Hansen (1992, 1996) and Garcia (1998) for Markov-switching models and AIC, BIC and HQIC information criteria.

[4] $I_{(d\le\tau)}$ and $I_{(d>\tau)}$ refer two dummy variables :

$$I_{(d\le\tau)} = \begin{cases} 1 & if\ d \le \tau \\ 0 & otherwise \end{cases} \quad and \quad I_{(d>\tau)} = \begin{cases} 1 & if\ d > \tau \\ 0 & otherwise \end{cases}$$

According to the duration models, the probabilities p_{ij}^{d} and p_{ik}^{d} refer to hazard functions. They reflect the instantaneous probability to change from the regime *i* to regimes j and *k* given that the regime *i* has spent d periods. Nevertheless, it is possible that there is no a regime change during other periods. The later situation which appoints the regime persistence is considered as a possible destination:

$$p_{ii}^{d} = 1 - p_{ij}^{d} - p_{ik}^{d} \tag{1.4}$$

The effect of duration on the hazard functions (or transition probabilities) is especially recapped by the coefficients λ_2^{ij} and λ_2^{ik}. If one of these coefficients is significantly different from zero the hazard functions depend to the seniority of regime i. This dependence is so-called positive (resp. negative) when these coefficients are more positive (resp. negative). In this regard, the hazard functions are increasing (resp. decreasing) depending on the age of regime i. Furthermore, the level of persistence in the regime i is decreasing (resp. increasing) with duration. By assumption, this level becomes constant beyond the horizon of τ periods. According to Durland and McCurdy (1994) and Maheu and McCurdy (2000), the parameter τ refers to the memory of duration dependence[5].

Formally, our DD(τ)-MS(3)-AR(L)- DD(τ)-GARCH-M(1.1) model[6] may be written as :

$$R_t(S_t) = \alpha^{(S_t^*)} + \delta^{(S_t^*)} \ln\left(d_{S_t^*}\right) + \sum_{l=1}^{L} \beta_l^{(S_{t-l}^*)}\left[R_{t-1} - \alpha^{(S_{t-l}^*)} - \delta^{(S_{t-1}^*)} \ln\left(d_{S_{t-1}^*}\right)\right] + \eta^{(S_t^*)} h_t(S_t) + \varepsilon_t \quad , \tag{1.5}$$

$$h_t(S_t) = \gamma_0^{(S_t^*)} e^{\gamma_1^{(S_t^*)} \ln\left(d_{S_t^*}\right)} + \gamma_2^{(S_{t-1}^*)} (\tilde{\varepsilon}_{t-1}^{(S_{t-1}^*)})^2 + \gamma_3 \tilde{h}_{t-1} \quad , \tag{1.6}$$

$$\tilde{\varepsilon}_{t-1}^{(S_{t-1}^*)} = R_{t-1} - E_{t-2}\left[R_{t-1} / S_{t-1}^*\right], \tag{1.7}$$

$$E_{t-2}\left[R_{t-1} / S_{t-1}^*\right] = \frac{\sum_{n=1}^{N} E_{t-2}\left[R_{t-1} / S_{t-1} = n, Y_{t-2}\right] \Pr\left(S_{t-1} = n / Y_{t-1}\right) I_{(S_{t-1}^*)}}{\sum_{n=1}^{N} \Pr\left(S_{t-1} = n / Y_{t-1}\right) I_{(S_{t-1}^*)}} \quad , \tag{1.8}$$

$$\tilde{h}_{t-1}^2 = \sum_{n=1}^{N} h_{t-1}^2(n) \Pr\left(S_{t-1} = n / Y_{t-1}\right) \quad , \tag{1.9}$$

and

$$\varepsilon_t \rightarrow NID\left(0, h(S_t)\right),$$

where:

- R_t: Market index return at time t;
- Y_t : $(R_t, R_{t-1}, \ldots, R_1)$, the market index return's vector for periods t, t-1, ..., 1;

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5 Given that the parameter τ can take only discrete value, it is chosen (i.e. grid search method) to maximize the log-likelihood function starting from $\tau_{min} = l + 1$.

6 A statistical approach is used in order to justify the relevance of adopted specification. First, the series of TUNINDEX index weekly returns was tested for stationarity by applying Augmented Dickey-Fuller and Phillips-Perron test. To determine secondly the order of AR(p) process to depict the dynamic behavior of returns through AIC and BIC information criteria. The actual values of the autocorrelations and partial autocorrelations and the Ljung-Box test results for filtered weekly series (i.e. the series of residuals provided by adjusting the series of returns by ARIMA model) are thirdly used. Finally, the Lagrange multiplier test is established to analyze ARCH effect in the filtered series. The choice of the three states is justified a posteriori using the Likelihood-ratio test proposed by Hansen (1992, 1996) and Garcia (1998), information criteria and MSC criterion for Markov-switching models.

- S_t: {1, 2, 3, ..., N}, a "N states" latent variable, according to a first-order Markov chain characterized by the following property :

$$\Pr(S_t = n/S_{t-1} = m, \dots, Y_{t-1}) = \Pr(S_t = n/S_{t-1} = m, Y_{t-1}) = p_{nm}$$

n, m = {1, 2, 3, …, N} ; (1.10)

- $\Pr(S_t = n / Y_{t-1})$, n = {1, 2,…, N}, smoothed probabilities' vector, which expressed, for any time t, the unconditional probabilities of states occurrence, knowing all the information up to time t-1.

Each realization of the variable S_t refers to a possible (L + τ) length trajectory of DD(τ)-MS(3)-AR(L)-DD(τ)-GARCH-M(1.1) process. By construction, the path is identified to a vector of weekly situations depicting the S_t^* and $D_{S_t^*}$ variables dynamics over the temporal spells of (L + τ) periods (week). The variable $I_{(S_t^*)}$ indicated the paths achieving to the same situation S_t^* at the end of spell[7].

The equation (1.5) above indicates the market index return' evolution that is governed by an L order autoregressive process. In this paper, the Ljung-Box (1978) test is applied to the standardized residuals of the equation (1.5) to pronounce on the L autoregressive order. The error terms of the return equation are unobserved because the regimes are unobservable. Therefore, according to Dueker (1997) and Maheu and McCurdy (2000), we retain for this test the standardized expected residuals[8]:

$$\sum_{J=1}^{N} \frac{R_t - E\left[R_t / S_t = n, Y_{t-1}\right]}{\sqrt{h_t(S_t)}} \Pr\left(S_t = n/Y_{t-1}\right) \qquad (1.11)$$

Moreover, we expressed the autoregressive terms of the volatility equation in line with the $\tilde{\varepsilon}_{t-k}$ and $\tilde{h}_{t-1}^{(S_{t-1}^*)}$ terms rather than the ε_{t-k} and $h_{t-1}^{(S_{t-1})}$, respectively.

Equations (1.5) and (1.6) indicated that the conditional mean and variance can change with duration. This provides us to analyze dynamic behavior for the mean and the variance within each state. The exponential form adopted in equation (1.6) is likely to ensure the positivity of the conditional variance. The equation (1.5) can elucidate the nature of the relationship between return and risk (i.e. risk-return trade-off) at each through the link between the conditional mean and variance.

Data

The data set, that will be used for empirical application, consists of weekly returns calculated from the Tunisian Stock Exchange, covering the period from 07/01/1998 to 29/03/2013. We thus have 931 observations.

Descriptive statistics for the data are provided in Table 1 while figure 1 plots the weekly returns for our sample.

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[7] Appendix I reports how to construct these paths and exhibits a maximum Likelihood procedure for estimating the parameters of the model.

[8] Under these conditions, the Ljung-Box (1978) test results can be used as an indication because the asymptotic distribution of the statistics is unknown.

Table 1. Descriptive Statistics of the TUNINDEX Weekly Return Data

	07 Jan 1998-23 Mar 2013
Mean	0.1668
Median	0.0864
Minimum	-10.9427
Maximum	9.3009
Standard Deviation	1.38022
Skewness	-0.0749 (0.080)
Kurtosis	12.4102 (0.160)***
Jarque-Bera	5905.241
Probability	0.000000

Notes : - Summary statistics for TUNINDEX index returns from 07/01/1998 to 23/03/2013;
- Standard errors are displayed as (.); -***: Significance level at 1%.

As reported in table 1, the TUNINDEX index weekly returns vary between -10.9427% and 9.3009%. The mean weekly TUNINDEX return is 0.1668% and the return standard deviation is 1.3802%. These descriptive statistics show high volatility which marks the TUNINDEX index weekly evolution. These values are also a reflection of strong stock *price* fluctuations on the Tunisian stock market[9].

As a result of the excess volatility, the return distribution for TUNINDEX index is characterized by a large number of extreme values. Figure 1 reports the boxplot for return distribution. At first sight, figure 1 displays mainly three blocs which allow us to identify a priori three different states. This suggests also the non-normality of the data set. The TUNINDEX index weekly returns are leptokurtic and exhibit fat tail phenomenon. The Jarque–Bera statistic points out the departure from normality for all return series at the 1% level of significance.

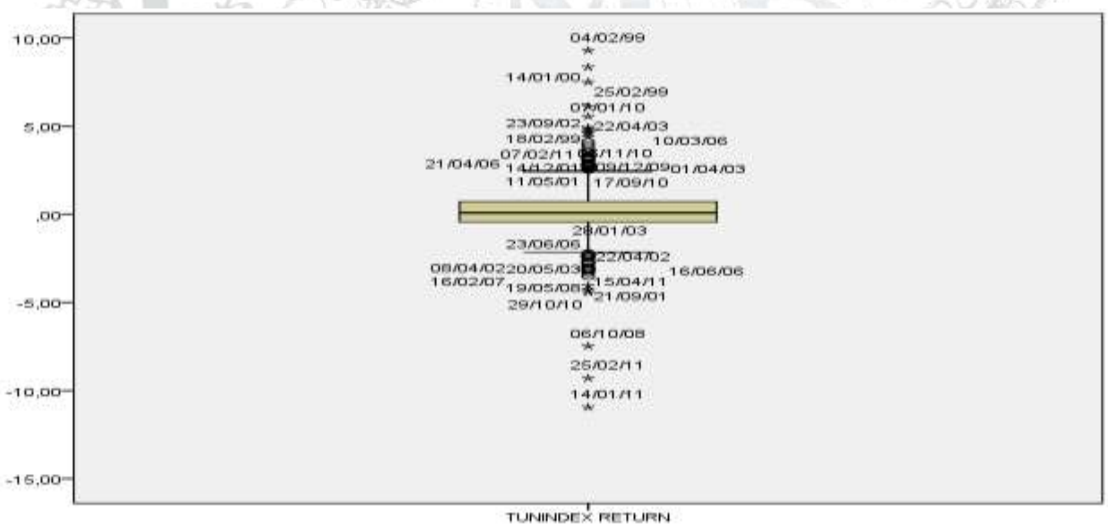

Figure 1. Boxplot

To examine the linear dependence of stock returns, we use the modified Ljung-Box Q statistic. The Q statistic is significantly different from zero for different lags. These results indicate the presence of the linear autocorrelations in TUNINDEX index returns series. Besides, the adequacy of the model

[9] This apparent feature of the data set is illustrated thereafter from the green curve reported in figure 5.

specification is attested through analysis of disturbances in the observable variables. We apply the BDS test to filtered data (i.e. the residual series estimated from ARMA model)[10] in order to examine the presence of nonlinear dependence in the Tunisian index returns series. The BDS statistic is significantly different from zero for different values of m and ε. This shows the nonlinearity in the returns. Finally, the value of Lagrange multiplier statistic of 362.209 is beyond 6.83 (the critical value of $\chi^2(1)$ at the significance level of 1%). This displays the ARCH effect in the filtered series. The GARCH models may be more appropriate for modeling volatility.

Estimation Results

To search adequate specifications for reproducing the TUNINDEX index's nonlinear dynamics over 07/01/1998-29/03/2013, we examine four models which differ mainly by the number of regimes. To progress, we first estimated single-regime model (M1) with AR(2) specification for the mean return equation and a GARCH(1,1) specification for the volatility equation. In this paper, we consider the model M1 as the benchmark model. The coefficients estimated by the maximum likelihood method are provided in the second column of Table 3.The estimates of coefficients' standard deviations are given in parentheses. The significance of all coefficients leads us immediately to reject the random walk hypothesis in order to describe the evolution of the TUNINDEX index weekly returns' fluctuations. In particular, the significant level of β_1 coefficient (related to the part of autoregressive GARCH process) already gives us the conviction of nonlinear process. Given a quadratic parameterization of the conditional variance, the GARCH type specification is limited uniquely to study the non-linearity in the volatility equation. In this context, the multistate models that analyze jointly the non-linearity effects on the mean and the volatility are: the two-state DD-MS-GARCH-M model (M2), the three-state DD-MS-GARCH-M model (M3) and the four-state DD-MS-GARCH-M model (M4).

Determination of the Number of States

Determining the number of states poses generally substantial difficulties, yet is clearly important to understanding the properties of the stock return process. To select a model specification, we use standard information criteria (AIC, BIC, HQIC) and the Likelihood Ratio test. We also employ the Markov Switching Criterion (MSC) proposed by Smith et al. (2006). The MSC criterion is based on the divergence of Kullback-Leibler to choose the number of states in the multi-state Markov-switching model.

Table 2. Log-likelihood, Information Criteria and Likelihood Ratio Test for Different Models

Model[g]	lgl[a]	k[b]	AIC[c]	BIC[d]	HQIC[e]	MSC	LR test[f]
M1	-975.167	5	2.1056	2.1316	2.1155	1953.334	-
M2	-787.255462	21	1.7363	1.8454	1.7779	2512.511	375.823
M3	-195.918109	37	0.5004	0.0112	0.5737	1339.836	1182.675
M4	NC	51	-	-	-		-

Notes: - (a): Log-likelihood value;
- (b): Number of parameters;
- (c), (d) and (e): The Akaike, Schwartz and Hannan-Quinn information criteria, respectively;
- (f): Likelihood ratio test;
- (g):The model M1 refers to single-state model with an AR(2) specification for the mean return equation and a GARCH-M(1,1) specification for the volatility equation.

[10] The Box-Pierce test results display that we can limit to the second-order autoregressive model in order to estimate no-correlated residuals.

The models M2 and M3 pertain to the two-state and three-state models, respectively. For reasons of misspecification resulting in the non-convergence of the estimation algorithm, the model M4 did not hold in this paper.

We suggest that at least two states and probably three states are needed to model asset returns. Consistent with this we found that a single-state model was systematically rejected in likelihood ratio test. To further support our choice of model we considered the AIC, HQIC and BIC information criteria and found that a three-state specification is suitable for the weekly return series. The lowest value of MSC for three-state model indicates that the three-state specification was preferred.

Interpretation of the States

Estimates for our proposed three-state DD-MS-GARCH-M model are provided in the last column of table 3. Estimates of the parameters as well as the coefficients of transition probabilities equation (i.e. eq. 1.2 and 1.3) is obtained by means of the method of maximum likelihood.

Table 3. Estimation Results of the Models

Parameters		Benchmark Model	DD(τ)-MS(3)-AR(2)- DD(τ)-GARCH-M (1,1) Model
Return Equation	$\alpha^{(1)}$ $\alpha^{(2)}$ $\alpha^{(3)}$	0.0018 (0.0032)***	0.9323 (0.0201) *** -2.1525 (0.0315) *** -0.6733 (0.0205) ***
	$\beta_1^{(1)}$ $\beta_1^{(2)}$ $\beta_1^{(3)}$	-0.0160 (0.0017)***	-0.1627 (0.0119) *** 0.0216 (0.0079) *** 0.2574 (0.0088) ***
	$\beta_2^{(1)}$ $\beta_2^{(2)}$ $\beta_2^{(3)}$	0.0422 (0.0128)**	0.6183 (0.0065) *** 0.6382 (0.0133) *** 0.3789 (0.0047) ***
	$\delta^{(1)}$ $\delta^{(2)}$ $\delta^{(3)}$		0.3349 (0.0146) *** -0.2095 (0.0679) *** 0.5523 (0.0135) ***
	$\eta^{(1)}$ $\eta^{(2)}$ $\eta^{(3)}$		0.9476 (0.0835) *** -0.1602 (0.0152) *** 0.8621 (0.0631) *
Volatility Equation	$\gamma_0^{(1)}$ $\gamma_0^{(2)}$ $\gamma_0^{(3)}$	0.0002 (6.6 E-06)**	0.0069 (0.0004) *** 0.9280 (0.1381) *** 0.0172 (0.0015) ***
	$\gamma_1^{(1)}$ $\gamma_1^{(2)}$ $\gamma_1^{(3)}$		-0.5525 (0.1419) *** -2.4196 (0.1665) *** -1.5026 (0.1658) ***
	$\gamma_2^{(1)}$ $\gamma_2^{(2)}$ $\gamma_2^{(3)}$	0.2929 (0.0213)**	0.0327 (0.0037) *** 0.8364 (0.0839) *** 0.0034 (0.0012) ***
	γ_3		0.0379 (0.0052) ***

Equation of Transition Probability	$\lambda_1^{(1,2)}$		-0.8415 (1.8343)
	$\lambda_1^{(1,3)}$		1.9922 (1.3744)*
	$\lambda_1^{(2,1)}$		1.8266 (0.3596)***
	$\lambda_1^{(2,3)}$		1.7719 (0.4111) ***
	$\lambda_1^{(3,1)}$		-1.2654 (0.6587)**
	$\lambda_1^{(3,2)}$		-6.7238 (0.9363) ***
	$\lambda_2^{(1,2)}$		0.3133 (0.144)**
	$\lambda_2^{(1,3)}$		-1.3762 (0.6081)**
	$\lambda_2^{(2,1)}$		-0.3098 (0.1442)**
	$\lambda_2^{(2,3)}$		-0.4713 (0.1847)***
	$\lambda_2^{(3,1)}$		1.6177 (0.4897) ***
	$\lambda_2^{(3,2)}$		-0.2895 (0.5991)
lgl		-975.167	-195.918109
N		931	931
T			12
Misspecification Tests			
Q(5)		5.426	3.315
Q(10)		10.125	8.736
Q(15)		16.464	12.197
Q(20)		20.754	19.532
Q(30)		41.736*	28.523

Notes : -* Significant at the 10% level; - ** Significant at 5% level; - *** Significant at 1% level;
- Standard errors are displayed as (.); lgl is the log-likelihood value; N is number of observations.

From table 3, the state 1 is characterized by increasing positive returns ($\alpha^{(1)}$ *and* $\delta^{(1)}$ are both significantly positive) whereas the state 3 delivers decreasing negative returns ($\alpha^{(3)}$ *and* $\delta^{(3)}$ are both significantly negative). The conditional return in state 2 is weakly negative ($\alpha^{(2)}$= -0.6733) which increases with duration ($\delta^{(2)}$ = 0.5523). Fig 2.1 plots the conditional return in different states against duration[11]. The difference in returns across different states is clearly pronounced. In particular, state 2 is characterized by intermediate levels of return. By taking into account the conditional volatility in states, we found that different states are associated with positive variance ($\gamma_0^{(1)}$, $\gamma_0^{(2)}$ and $\gamma_0^{(3)}$ are significantly positive). Specially, state 2 has higher volatility ($\gamma_0^{(2)} = 0.9280$)than other states($\gamma_0^{(1)} = 0.0069$ and $\gamma_0^{(3)} = 0.0172$). When the conditional volatility is dependent on duration, the volatility in all three states decreases over time. Fig 2.2 plots the conditional volatility in states as duration raises. The three states show a negative dependence of the conditional volatility in states as duration increases. We found clearly empirical evidence of states with different features in terms of mean return and volatility by applying the three-state DD-MSAR-GARCH-M model to TUNINDEX index returns. State 1 exhibits increasing high returns and decreasing positive volatility while state 3 is

[11] Note that the memory τ of duration dependence for the three-state DD-MS-GARCH-M model was determined to be 12.

characterized by decreasing low returns and decreasing positive volatility. State 2 is increasing weakly negative-return and decreasing positive-volatility state.

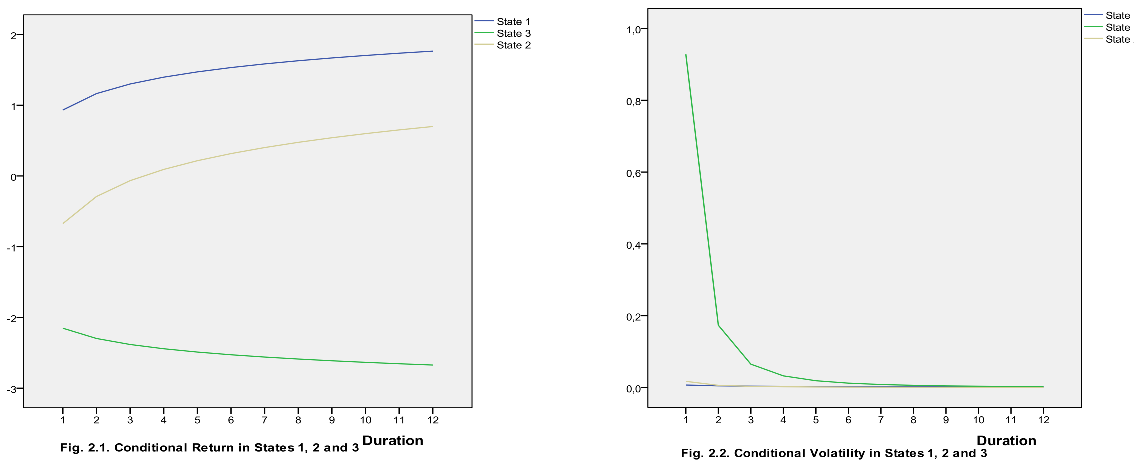

Figure 2. Conditional Return and Conditional Volatility from Three-state DD-MSAR-GARCH-M Model

Again, the relationship between risk and return is neither stable over time nor linear (e.g. Lo, 2004). From table 4, the parameters $\eta^{(i)}$ (i = 1,2,3) which refer to the risk-return trade-off are all significantly different from 0 (resp. equal to 0.9476,-0.1602 and 0.8621 for bull, bear and normal states). The risk-return trade-off increases (resp. decreases) with the length of time spent in the states 1 and 2 (resp. the state 3).

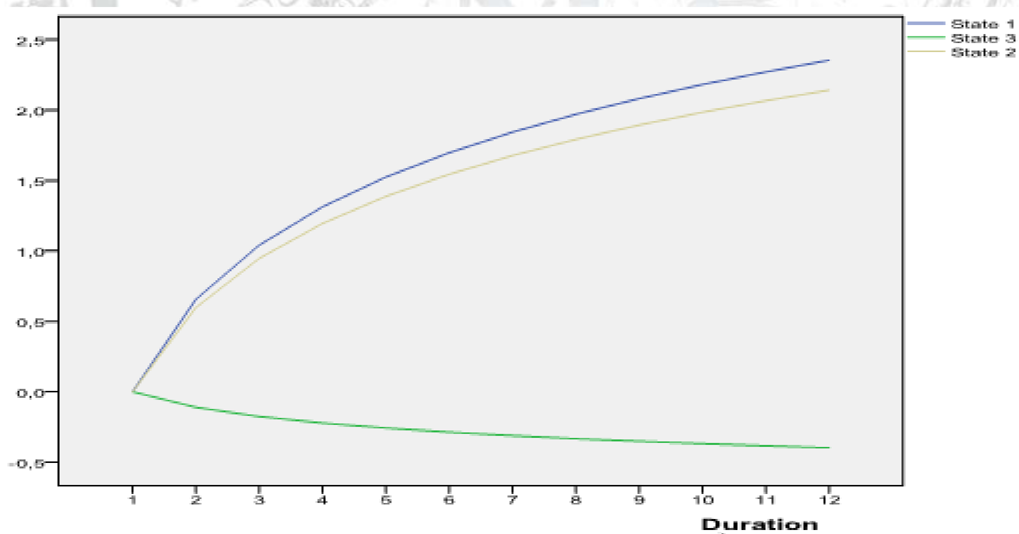

Figure 3. Risk-Return Trade-off in States 1, 2 and 3

Looking thereafter at the results for the duration-dependent transition probabilities, we find that differences in the sign, amplitude and level of significance of $\lambda_2^{(i,j)}$ and $\lambda_2^{(i,k)}$ lead us to different interpretations. The implications for transition dynamics between states are summarized in figure 4. This shows how the probability of moving from one state to another as duration raises. Three plots in figure 4, i.e. Fig 4.1, 4.5, 4.9, refer to persistence in the states 1, 2 and 3, respectively. In contrast, the other six plots illustrate the migration from one state to another.

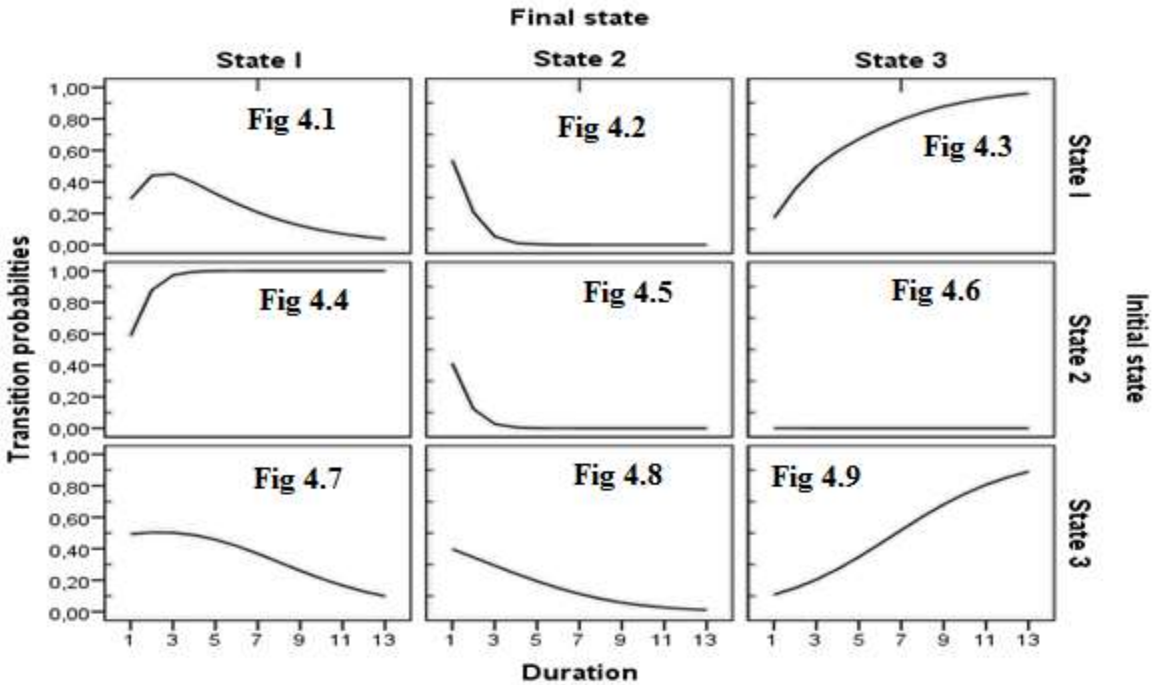

Figure 4. Three-state DD-MS-GARCH-M Model : Transition Probabilities

At the first sight, we note that the each curve's shape is either increasing or decreasing according to the age of regime (expressed in weeks). The nonlinear behavior in the transition probabilities displays a priori the importance of the length of time spent in the different market states. As plotted in figures 4.1, 4.2 and 4.3, during the first week, the state 1 seems generally to persist or migrate to the states 1 and 2 with respective probabilities of about 0.3, 0.55 and 0.15. Nonetheless, the probability of staying in the state 1 increases to reach about 0.45 during the first four weeks. Beyond this period, the persistence of the state 1 diminishes gradually as duration increases. Fig 4.2 which depicts the decay of migration from the state 1 to the state 2 shows the decline of chance to entry in the state 2. As a matter of fact, the probability of moving from the state 1 to the state 2 is equal to zero at the end of the fourth week. On the other side, the longer the duration the higher is the probability of migration from the state 1 to the state 3 (Fig 4.3).

The situation is quite different when the market is in state 2. That is, it appears to be about 40% likely to persist and 60% to migrate to the state 1 at the first week. However, the probability of staying in the state 2 declines as duration increases (Fig 4.5). From Fig 4.6, the migration from the state 2 to the state 3 is almost unlikely.

When the Tunisian stock market is in the state 3 the probability of moving from the state 3 to the state 1 reaches about 50% during the first five weeks (Fig 4.7). Beyond the first three weeks, the probability of transition from the state 3 to the state 1 decreases with duration. As a matter of fact, Fig 4.9 shows clearly this evidence. Staying into the state 3 seems more pronounced as duration increases. Likewise, Fig 4.8 which illustrates the migration from the state 3 to the state 2 displays the stock market rally during the second week. Nevertheless, this situation does not persist as duration increases. On average, the stock market spent 50% of time in the state 1 and 5% in the state 3. The unconditional probabilities of the states 1, 2 and 3 are 50%, 45% and 5%, respectively. In fine, these are the unconditional probabilities for states 1, 2 and 3, i.e. $p(S=i)=\sum_{d=1}^{\tau} p(S=i, D=d), i=1,2,3.$

Turning Index for Probabilistic Lecture of Tunisian Stock Market Cycle

Apart from the duration-dependence and nonlinearity issues discussed above, the three-state model offers mainly the bases for a probabilistic study of the Tunisian stock market cycle. This allows us to highlight the cyclical fluctuations of TUNINDEX index while tracing the paths according to which returns chronicles is described by a succession of downward, upward or no trend phases. Recall that the dating algorithm (rules-based approach) developed by Bry and Boschan (1971) and regime-switching models attempt both to identify economically significant regimes. Our DD(τ)-MS(3)-AR(2)-GARCH-M(1,1) model-based procedure is implemented using again weekly TUNINDEX stock index data for 07 January 1998 through 29 March 2013.

Like Bellone et al. (2005), the estimation of smoothed probability allows us to construct an indicator which varies between -10% and +10%. This indicator, so-called turning indicator (or turning index), is computed as the difference between the smoothed probabilities of being in the bull and bear states[12].

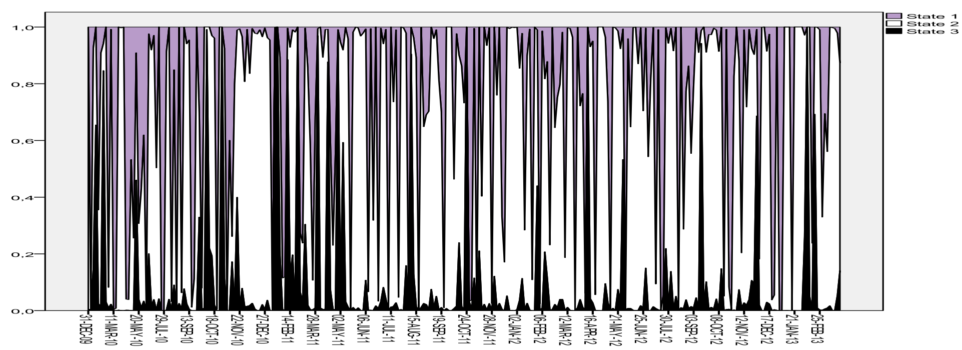

Figure 5. Smoothed State Probabilities: The Three-State Model for Stock Returns from 31/12/2009 to 29/03/2013

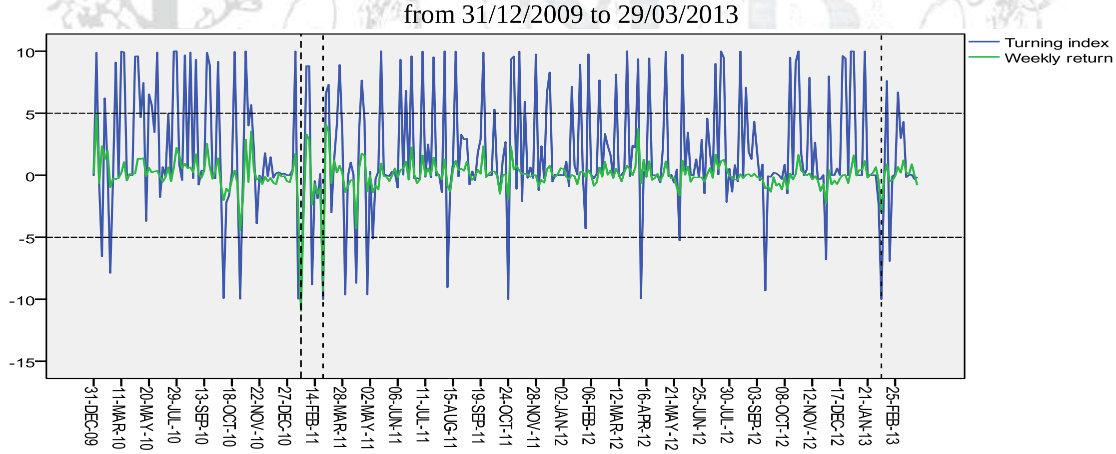

Figure 6. Turning Index of Tunisian Stock Market Cycle[13] from 31/12/2009 to 29/03/2013

[12] Formally, we propose Turning Index (TI_t) is given as follows : $TI_t = 10 * (P[S_t = 1/Y_T] - P[S_t = 2/Y_T])$.
where $p(S_t = i|Y_T)$ $(i = 1,2,3)$ represent the smoothed probabilities. Each observation at time t is conferred to particular regime i if $p(S_t = i/Y_T) > p(S_t = j/Y_T)$ $(i, j = 1,2,3)$. Thus, $P[S_t = 1/Y_T]$ is the probability of being in the bull state whereas $P[S_t = 2/Y_T]$) is the probability of being in the bear state.

[13] For a better legibility of the graph, we restrict the presentation to last four years: 2010-2011-2012-2013 (i.e. 229 observations).

Figure 5 plots the state probabilities for the three-state DD-MS-GARCH-M model while figure 6 illustrates the evolution of turning index as well as TUNINDEX stock index weekly return during 07/01/1998-29/03/2013. According to our turning index, belonging to the bull state (resp. bear state) is indicated by values greater than 5% (resp. lower than -5%); the values between -5% and 5% referred to the normal state[14]. By applying this rule, we can decide on the different market states in a probabilistic way throughout the observation period.

The relevance of this approach seems to be well documented from the synchronization, as can be seen from figures 5 and 6, between the values of the turning index on the one hand and the values of TUNINDEX index return on the other hand. In particular, our proposed index return is well-adapted to account for extremes events. So, for the date of January 14, 2011, the turning index recorded a value of -10%, indicating an exceptional fall of TUNINDEX index (about -11%) due to events related to the Tunisians' uprising against the ousted president. Likewise, at February 25, 2011, the turning index value of about 10% accounts for a second important drop of TUNINDEX index (equals about 9.30%). In this respect, the stock market authorities were forced to suspend the trading activity in order to deal with this crisis situation. Moreover, the turning index was dropped during the day of February 06, 2013 (the third dashed line) to highlight the distress in the Tunisian stock market following the news of opposition leader Chokri Belaid's assassination.

Concluding Remarks

In this paper, we proposed a probabilistic lecture of Tunisian stock market cycle based on turning index from the estimation of three-state model. In this respect, we compared three different models: the single-regime model, the two-state DD-MSAR-GARCH-M model (M2) and three-state DD-MSAR-GARCH-M model (M3) using information criteria, MSC criterion and Likelihood ratio test. Results point to a three-state specification for the weekly return series.

Estimates results suggest that the three-state model is accordance with the interpretations often suggested in the bull and bear literature (e.g. Chauvet and Potter, 2000; Maheu and McCurdy, 2000). As highlighted previously, the state 1 is characterized by increasing positive returns whereas the state 2 delivers decreasing negative returns. According to the Chauvet and Potter (2000)'s definition[15], we can now qualify the state 1 as bull state while the state 2 is the bear state. The state 3 is a 'regime of center' characterized by intermediate levels of return. The normal market label could refer to state 3.

To sum up, the main empirical facts established by this paper can be described succinctly. The fluctuations of the Tunisian stock market should be thought of as having three phases rather two: speculative excess phase, normal phase and high-growth recovery period following crash.

The risk-return trade-off increases (resp. decreases) with the length of time spent in the states 1 and 2 (resp. state 3). This evidence reflects necessarily the investors' changing risk attitude with the amount of time spent (i.e. duration) in a particular market state. During decreasing positive-volatility state (state 1), investors appreciate the low volatility and become more risk takers implying the increase of their demand for the shares. Because the relationship between risk and return increases over time in state 2, we still have risk preference behavior in the stock market. In contrast, state 3 which characterized by high volatility implies the smaller holding of risky assets because investors are risk averse.
By focusing on the duration-dependent transition probabilities, the three-model offers new substantive insights about states. Starting from the state 1, which provides very attractive returns, we highlight the three

[14] The choice of the two limits of -5% and +5% is not arbitrary. These values are obtained from the optimal binning method which divides a variable into small number of intervals (or bins) with respect to a categorical guide variable.
[15]According to Chauvet and Potter (2000), the bull market (resp. bear market) corresponds to periods of generally increasing (resp. decreasing) market prices.

different ways that the stock market can end up state 1. Most of the time, the speculative excess caused by the willingness of investors to profit from the short term fluctuations in the stock market and pursue the abnormal returns lead to staying in state 1. As we have seen in Fig 4.1, the speculative movement in the stock market which spent only about three weeks is followed by the fall in stock prices. Fig 4.3 illustrates clearly this evidence. The trend reversal which is change in the direction of the stock market (from state 2 to state 1) also allows to achieve the state 1. Two foreseeable factors are behind the trend reversal in the stock market. The mimic attraction of large number of investors leads to the herding behavior implying the moving to state 1. Here it is important to note that the desire to imitate observed decisions of others investors calls into question the speed of the information diffusion in the stock market and the investors' processing capacity for new information. Another possible factor for the swing from state 2 to state 1 is the momentum effect or positive feedback trading in the stock market. Once investors observe an acceleration in stock price by using generally the technical analysis, they become more optimist and overconfident and attempt to increase their stock investment turning into buying panics. State 2 thus seems to be a transitory phase which reflects the natural evolution of Tunisian stock market.

Fig 4.7 provides us another piece of evidence that the stock market tends to quickly bounce back up following the downturn. In the same way as in the business cycle, contraction in the stock market activity is generally succeeded by relatively short period of very high growth. As matter of fact, the volatility in state 3 is high during the first week. As duration increases, the bounceback feature does not persist and the selling panic to avoid losses pushes down the stock prices. As a result, the stock market plunges in the crisis situation (i.e. crash). Therefore, state 2 encompasses two subphases: first a recovery of the stock market to increasing high-return state (state 1) and then a period of the stock prices collapse.

Finally, we propose a probabilistic index based on the smoothed probabilities of the three-state model to characterize the different Tunisian market cycle from 31/12/2009 to 29/03/2013. Results emphasize the synchronization between the values of the turning index and the values of TUNINDEX index return. Our turning index also allows to highlight the extreme events.

Appendix

All the parameters of the model described in section 1 are jointly estimated through the maximum likelihood method. The likelihood function itself is evaluated recursively using formula to that in Hamilton (1989) and Kim (1994). The discussion below is a Durland and McCurdy (1994), and Maheu and McCurdy (2000) procedure generalisation to estimate a three-state Duration Dependent Markov Switching model ((DDMS-3).

Given a three-state l-lag model with two sates variables, S and D, we specifically define the N-states vector at time t, Σ_t, in this way:

$$\left|\begin{array}{lllll} S_t=1, & S_{t-1}=2, & \ldots, & S_{t-l}=1, & D(S_t)=1 \\ S_t=1, & S_{t-1}=2, & \ldots, & S_{t-l}=2, & D(S_t)=1 \\ S_t=1, & S_{t-1}=2, & \ldots, & S_{t-l}=3, & D(S_t)=1 \\ & & \vdots & & \\ S_t=1, & S_{t-1}=1, & \ldots, & S_{t-l}=1, & D(S_t)=l \\ S_t=1, & S_{t-1}=1, & \ldots, & S_{t-l}=1, & D(S_t)=l+1 \\ & & \vdots & & \\ S_t=1, & S_{t-1}=1, & \ldots, & S_{t-l}=1, & D(S_t)=\tau \end{array}\right|$$

$$\left|\begin{array}{lllll}
S_t=2, & S_{t-1}=1, & ..., & S_{t-l}=1, & D(S_t)=1 \\
S_t=2, & S_{t-1}=1, & ..., & S_{t-l}=2, & D(S_t)=1 \\
S_t=2, & S_{t-1}=1, & ..., & S_{t-l}=3, & D(S_t)=1 \\
 & & . & & \\
 & & . & & \\
 & & . & & \\
S_t=2, & S_{t-1}=2, & ..., & S_{t-l}=2, & D(S_t)=l \\
S_t=2, & S_{t-1}=2, & ..., & S_{t-l}=2, & D(S_t)=l+1 \\
 & & . & & \\
 & & . & & \\
 & & . & & \\
S_t=2, & S_{t-1}=2, & ..., & S_{t-l}=2, & D(S_t)=\tau
\end{array}\right|$$

$$\left|\begin{array}{lllll}
S_t=3, & S_{t-1}=1, & ..., & S_{t-l}=1, & D(S_t)=1 \\
S_t=3, & S_{t-1}=1, & ..., & S_{t-l}=2, & D(S_t)=1 \\
S_t=3, & S_{t-1}=1, & ..., & S_{t-l}=3, & D(S_t)=1 \\
 & & . & & \\
 & & . & & \\
 & & . & & \\
S_t=3, & S_{t-1}=3, & ..., & S_{t-l}=3, & D(S_t)=l \\
S_t=3, & S_{t-1}=3, & ..., & S_{t-l}=3, & D(S_t)=l+1 \\
 & & . & & \\
 & & . & & \\
 & & . & & \\
S_t=3, & S_{t-1}=3, & ..., & S_{t-l}=3, & D(S_t)=\tau
\end{array}\right|$$

Where $N=3^{l+1}+3(\tau-l-1)$. Let $Y_t = (R_t, R_{t-1},\ldots, R_1)$ denote all observations obtained through date t, and let θ denote the entire parameter set of the model. The input for step t of the filter is the conditional probability vector of Σ_t (i.e. $\Pr(\Sigma_t / Y_{t-1;}\ \theta)$), and the output is the full-sample smoother probabilities vector:

$$\tilde{\Pi}_t = \Pr\left(\Sigma_t \ / \ Y_T \ ; \theta\right);$$ T is the number of the full sample periods.

The vector of unconditional probabilities of Σ_t, $\Pi_t(\theta) = \Pr(\Sigma_t / Y_{t;}\ \theta)$, is (N x 1) and the associated transition matrix, $P = \Pr(\Sigma_t / \Sigma_{t-1})$, is (N x N). This later is defined as

$$P = \begin{bmatrix} P_{11} & P_{21} & . & . & . & P_{N1} \\ P_{12} & P_{22} & . & . & . & P_{N2} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ P_{1N} & P_{2N} & & & & P_{NN} \end{bmatrix},$$

where: $P_{ij} = \Pr\left(\Sigma_t^{(j)} \ / \ \Sigma_{t-1}^{(i)}\right)$; $\Sigma_t^{(j)}$ and $\Sigma_{t-1}^{(i)}$ are the *j*th and the *i*th rows of Σ_t and Σ_{t-1} respectively. Each P_{ij} is constructed from the conditional probabilities in equations (3) and (4) and the (N x N) transition matrix can be computed as follows:

For j = 1, 2.

The basic filter proceeds then as follows:

1. Calculate

$$\Pr(\Sigma_t / Y_{t-1} ; \theta) = \Pr(\Sigma_t / \Sigma_{t-1}).\Pr(\Sigma_{t-1} / Y_{t-1} ; \theta).$$

2. Calculate the joint conditional density distribution of R_t and Σ_t:

$$f(R_t, \Sigma_t / Y_{t-1}; \theta) = f(R_t / \Sigma_t, Y_{t-1}; \theta) \otimes \Pr(\Sigma_t / Y_{t-1}; \theta)$$

where the symbol $\otimes$ denote element-by-element multiplication, and

$$f(R_t / \Sigma_t, Y_{t-1}, \theta) = \begin{bmatrix} f(R_t / \Sigma_t^{(1)}, Y_{t-1}, \theta) \\ f(R_t / \Sigma_t^{(2)}, Y_{t-1}, \theta) \\ . \\ . \\ . \\ f(R_t / \Sigma_t^{(N)}, Y_{t-1}, \theta) \end{bmatrix}$$

3. We then have

$$f(R_t / Y_{t-1}; \theta) = \iota'(f(R_t / \Sigma_t, Y_{t-1}, \theta) \otimes \Pr(\Sigma_t / Y_{t-1}; \theta))$$

ι is a (N x 1) vector of ones.

4. The zero-lag smoother can then be obtained from the filter by dividing the relevant elements in (2) the scalar in (3):

$$\Pi_t(\theta) = \Pr(\Sigma_t / Y_t; \theta) = \frac{f(R_t, \Sigma_t / Y_{t-1}; \theta)}{f(R_t / Y_{t-1}; \theta)}$$

A by-product of the basic filter in (3) is the likelihood function

$$L(\theta) = \prod_{t=1}^{T} f(R_t / Y_{t-1}; \theta)$$

Based on Hamilton (1994), the model is estimated by setting $\Pr(\Sigma_1 / Y_1; \theta)$ to equal the unconditional probabilities (i.e. we solve the unconditional probabilities π as the solution to P'π = π, subject to ι'π = 1 ; ι is a (N x 1) vector of ones).

The log-likelihood function can be maximised numerically with respect to the parameter vector θ. In this model, the memory of the markov processes is designed by a discrete-value parameter, τ, which can be determined using a grid search starting from $\tau_{min} = l+1$ to maximize the log-likelihood function.

5. Ones the parameter vector θ is estimated, we can also calculate the full-sample smoother probabilities vector. In contrast to $\Pi_t = \Pr(\Sigma_t / Yt; \theta)$, which conditions only on information over the interval [1, t], the full sample probabilities use information over interval [1, T], so may be defined as $\tilde{\Pi}_t = \Pr(\Sigma_t / Y_T; \theta)$. It is obtained using the following algorithm developed by Kim (1994):

Calculate

$$\tilde{\Pi}_t(\theta) = \Pr(\Sigma_t / Y_T; \theta) = \Pr(\Sigma_t / Y_t; \theta) \otimes \{\Pr(\Sigma_{t+1} / \Sigma_t)' \times [\Pr(\Sigma_{t+1} / Y_T; \theta)(\div)\Pr(\Sigma_{t+1} / Y_t; \theta)]\}$$

Where the sign (÷) denote element-by-element division. The smoothed probabilities can be found by iterating on (5) backwards for t = T-1, T-2, …, 1. The starting input for the full-sample smoother, $Pr(\Sigma_T / Y_T; \theta)$, is obtained from the last iteration of the filter.

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